David Mwakima

Berkeley on Infinite Divisibility

draft of May 30, 2020

The Mathematicians think there are insensible lines, about these they harangue, these cut in a point, at all angles these are divisible ad infinitum. We Irish men can conceive no such lines. George Berkeley, *Philosophical Commentaries* (1708 - 1709)

Abstract

Berkeley, arguing against Barrow, claims that the infinite divisibility of finite lines is neither an axiom nor a theorem in *Euclid The Thirteen Books of The Elements*. Instead, he suggests that it is rooted in ancient prejudice. In this paper, I attempt to substantiate Berkeley's claims by looking carefully at the history and practice of ancient geometry as a first step towards understanding Berkeley's mathematical atomism.

Keywords: Berkeley, infinite divisibility, continuity, incommensurables, Pythagoreans, Aristotle

1 Introduction

Writing in the fourth century BCE, Aristotle wrote in *Physics* 200b15-200b21,"What is infinitely divisible is continuous;" and in *Physics* 207b16-207b21 the converse, "What is continuous is divi[sible] ad infinitum."[1]The Aristotelian view that magnitudes are infinitely divisible was endorsed by Isaac Barrow in the mid seventeenth century. Barrow was the first Lucasian Professor of Mathematics at Cambridge, a post later held by his student Isaac Newton.[2] In Lecture IX of his *Mathematical Lectures* he said:

> There is no part in any kind of magnitude, which is absolutely the least. Whatever is divided into parts, is divided into parts which are again divisible...whatsoever is continued is always divisible into parts again divisible. I am not ignorant, how difficult this doctrine is admitted by some, and entirely rejected by others.

Berkeley, who had certainly read Barrow, in the early eighteenth century CE was one of the dissidents. He wrote:

[1]All references to Aristotle in what follows are taken from the translations contained in Aristotle (1984).

[2]For Barrow's view I have used Barrow (1734) *Mathematical Lectures* published posthumously in 1683.

> The infinite divisibility of finite extension, though it is not expressly laid down[3], either as an axiom or theorem in the the elements of that science, yet is throughout the same everywhere supposed, and thought to have so inseparable and essential a connexion with the principles and demonstrations in geometry, that mathematicians never admit it into doubt, or make the least question of it. *A Treatise Concerning the Principles of Human Knowledge* (PHK, henceforth) §123 (W2: 99)[4]

In this paper, I want to unpack this debate between Barrow and Berkeley. There are a lot of philosophical issues that come up in this debate. So in order to keep my paper within a reasonable length, I will only focus on evaluating Berkeley's claim in his actual published work i.e. the claim from PHK §123 that infinite divisibility is not an axiom or theorem of Euclidean geometry.[5] In other work (Mwakima, 2020), I discuss Berkeley's mathematical atomist thesis in more detail, the view that magnitudes are composed of indivisible points or *minima tangibilia* together with distance and ordering relations.[6]

My strategy for evaluating Berkeley's claim is based on a clue given by Berkeley in *The Philosophical Commentaries* (PC, henceforth) 263 W1:33:

> To Enquire most diligently Concerning the Incommensurability of Diagonal & side. whether it Does not go on the supposition of unit being divisible ad infinitum, i.e of the extended thing spoken of being divisible ad infinitum (unit being nothing also V. Barrow Lect. Geom:). & so the infinite indivisibility deduc'd therefrom is a petitio principii. PC 263 W1:33

This clue suggests going back to Heath (1956)'s *Euclid The Thirteen Books of The Elements* (*The Elements*, henceforth) in order to look for any evidence that refutes or substantiates Berkeley's claim in PHK §123. I will argue that Berkeley is right in pointing out that infinite divisibility is neither an axiom nor a theorem in *The Elements*.[7] The view that

[3]In fact, Barrow (1734, pp. 153, 155) claimed that even though mathematicians rarely *openly* assume infinite divisibility, they *covertly* assume it. So it is a bit surprising that he went to such a great extent to defend it. This is what I discuss in what follows.

[4]I shall follow contemporary Berkeleyan scholarship abbreviations where, for example, 'W2:99' refers to volume 2 page 99 of Luce and Jessop (1957) *The Works of George Berkeley, Bishop of Cloyne.*

[5]See Jesseph (1993, Chap. 2) and Jesseph (2005) for how Berkeley's thought evolved from the *Philosophical Commentaries* to the PHK.

[6]This view, I argue has roots in Pythagoras, Epicurus and Gassendi. Gassendi is mentioned by Berkeley in *The New Theory of Vision*(NTV, henceforth) §75 (W1:200) and Epicureanism is mentioned by Berkeley in PHK §93 (W2:82) contra Jesseph's claim in Jesseph (1993, 67).

[7]For an opposing view see Jesseph (1993, 48 - 53) and Jesseph (2005, 278 - 284). My paper is not intended as a critical evaluation Jesseph's view, although along the way I identify the ways in which I

magnitudes are infinitely divisible is a *philosophical thesis* due to Aristotle – it is neither an axiom in *The Elements* nor does it follow from *The Elements* Book I Proposition 10 (To bisect a given finite line). The upshot of my paper will be to show that Berkeley is right where he says:

> Ancient and rooted prejudices do often pass into principles; and those propositions, which once obtain the force and credit of a principle, are not only themselves, but likewise whatever is deducible from them, thought privileged from all examination. PHK §124

Anticipating what follows, my argument is that the long-held view of identifying infinite divisibility with continuity is a view due to Aristotle's idiosyncratic conception of mathematics. The idiosyncrasy will become clear when I compare the Pythagorean conception of mathematics with the Aristotelian conception of mathematics in section 3 below. In the claim in PHK §123, Berkeley was seeking a conception of geometry that was more faithful to *The Elements* than was Aristotle's conception. The faithfulness consisted in abandoning Aristotle's philosophical thesis of infinite divisibility and *potentially* existing points, and adopting the alternative view of construing geometrical magnitudes as composed of *actual* points or what Berkeley called geometrical minima.[8]

Together with the textual evidence from *The Elements* and evidence from commentators on *The Elements* like Proclus (1970) and Heath (1956), the soundness of my argument depends on whether I am justified in claiming that Aristotle's appeal to infinite divisibility is (1) based on an alternative conception of mathematics; and (2) a failure by Aristotle to distinguish between mathematical atomism from physical atomism, the view that all there is are atoms and void.[9] I will not be concerned with (2) in this paper.[10] The focus of my

differ. There is a lot that I have learned from Jesseph and there is a lot in his view that I agree with. My intention is to open up the possibility for fruitful debate regarding these matters by offering the historical and philosophical background that could have influenced Berkeley's philosophy of mathematics.

[8]In Mwakima (2020) I develop this idea. In that paper I argue that the geometrical minima of Berkeley are actual points with the canonical two place distance function d and order. d satisfies, for example, that $d(x, y) = 0 \leftrightarrow x = y$ for any two points x, y. Existing discussions in the literature such as Jesseph (1993, 58) appear to me mistaken on their evaluation of Berkeley. There is, for instance, no careful discussion by Jesseph of the distinction between parts and points; and between distance and cardinality. As a result Jesseph confuses Berkeley's correct claim that for any two points x, y in a finite line such that $x < y$, the Euclidean distance $d(x, y) = y - x$ or parts between these two points is some finite *number* of *units of distance*, with the incorrect claim that a finite line has a finite number of points.

[9]This claim has been made by Garber (1992, 123) and the sources cited there. Aristotle argues as if a mathematical thesis – the convergence of geometric series – must be true of the real world in order to solve Zeno's paradoxes. See the discussion by Heath in Heath (1956, Vol. 1, 233f.).

[10]See Mwakima (2020).

paper will be to defend (1) in order for my argument to be sound. With respect to this defense, I argue that it is Aristotle's conception of mathematics that led subsequent mathematicians, including Barrow, to misunderstand the upshot of the Pythagorean *number theoretic* discovery of the *geometrical thesis* that asserts the existence of incommensurable magnitudes – literally magnitudes that cannot be measured by a common measure – as alluded to by Berkeley in the clue from PC.[11]

The results of my paper would be of interest, I hope, not only to historians of philosophy, but also mathematicians interested in the history and philosophy of the continuum in general[12]; physicists interested in questions regarding the nature of the space-time manifold[13]; and philosophers interested in the history of Zeno's paradoxes and its modern formulation using the tools of measure theory.[14] Someone may say that the question of continuity of spatial figures or bodies is an empirical question and whether or not infinite divisibility of magnitudes is true cannot as far as we know be settled *a priori* or empirically.[15] We know that Cantor and Dedekind *postulated* the principle of continuity for spatial magnitudes.[16] Thus, it seems to be pointless to argue for or against infinite divisibility a priori. Nevertheless, while we might question whether there really is a right or wrong answer on the basis of reason alone (this I take to be the upshot of Kant's Second Antinomy[17]); this by itself does not mean that the question is philosophically uninteresting. What I hope to show in what follows is how intricate this question was and what mathematical and philosophical assumptions went into it – assumptions which Berkeley was evaluating a priori.

Here's how I have organized my paper. In the next section I introduce the terms which set the debate regarding infinite divisibility: how to conceive of points and how to understand the part-to-whole relation in *The Elements*. In section 3, I show how Aristotle's idiosyncratic conception of mathematics led him to reconceptualize continuity – a process which, among other things, culminated in the characterization of continuous quantities in terms of infinite divisibility. I also show how and why someone who ascribes to the

[11] See section 3 below for a more careful discussion.

[12] An up-to-date discussion can be found in Reeder (2018) together with the sources cited there and Kanamori (2020).

[13] What I have in mind is: (1) the issue between *gunkologists* (region-based conception of spacetime manifold) and *pointillists* (point-based conception of the spacetime manifold) views on the nature of the continuum. See Fano, Orilia, and Macchia (2014) for discussion; and (2) the assumption that space is continuous in continuum mechanics.

[14] What I have in mind is the work of among others Skyrms (1983), Sherry (1988) and Ehrlich (2014).

[15] See Maddy (1997, 143-157) for discussion regarding the validity of the assumption that space is continuous.

[16] See Heath (1956, Vol. 1, 234 - 237).

[17] See Friedman (1995) for Kant and infinite divisibility.

Pythagorean conception of mathematics together with the theory of proportions by Eudoxus can work without the notion of (potential) infinite divisibility in order to substantiate Berkeley's claim in PHK §123.

2 Points and Parts

2.1 Does infinitely many parts of a whole W imply that W is infinite?

Following his remark that the thesis of infinite divisibility is neither assumed nor proved in *The Elements*, Berkeley gives a surprising argument that what is infinitely divisible must contain infinitely many parts and consequently be infinitely large. I propose that we start here and work backwards to substantiate Berkeley's initial remark in PHK §123.

> If the terms extension, parts, and the like, are taken in any sense conceivable, [...] then to say a finite quantity or extension consists of parts infinite in number, is so manifest a contradiction. PHK §124 (W2: 99)

> [W]hen we say a line is infinitely divisible, we must mean a line which is infinitely great. PHK §128 (W2: 101)

There are really two arguments here although in the second quotation it is only implicitly implied by the talk of meaning. In the first argument Berkeley is arguing that from the supposition that a finite line (he uses 'extension') consists of infinitely many lines, it follows that the original line is infinite. But the original line is finite. Hence it follows by reductio that our supposition was wrong. The second implicit argument in the claim in the second quotation is that if a line is infinitely divisible, then we must we mean a line which is infinitely long. Since it is common ground that there are no infinitely long lines, it follows by reductio again, that our supposition was wrong.[18]

There are several proposals open to someone who wishes to reject Berkeley's argument. One may challenge the assumption that it was common ground in the seventeenth century

[18] I thank Jeremy Heis for fruitful discussion of these issues. Initially I had thought that Berkeley wanted to give a complicated or novel instrumentalist interpretation of infinite divisibility based on his doctrine of signs. I have come to see that was not his intention at all. In PHK §§125 - 128 *he is offering one possible explanation*, on the basis of his representative theory of generality, for what might have led the mathematicians, erroneously, to suppose the thesis of infinite divisibility. It does not mean that this explanation is the right one nor does it mean that this is the only way to construe the thesis of infinite divisibility or lead to its acceptance within mathematics. Cf. Jesseph (1993, 72 - 74) who thinks Berkeley is offering an instrumentalist account. I address this instrumentalist reading of Berkeley's philosophy of mathematics in Mwakima (2020).

that there are no infinite lines. In recent work[19], Schechtman has argued that Locke accepts the notion of an infinite "measure."[20] It is not clear on Schechtman's proposal whether the contrast between 'number' and 'measure' is supposed to be identical to the contrast between 'number' and 'magnitude' in Aristotle's account of quantity.[21] According to Schechtman, the notion of measure operative in Locke's account is the following: k is a measure of some quantity Q if and only if k is a mode (i.e. a "way of being") of Q and:

1. $k = n$ for some natural number n, or
2. For every quantity P whose measure is some natural n, $k > n$

What this means is that a measure is either a natural number (in the finite case), or a mode of quantity that is greater than any given natural number. On the basis of this, she proposes that Locke's notion of quantitative infinity is expressed by the formula below (where Fx stands for 'x is finite' and k and l are measures):

$$\exists k \forall l (Fl \rightarrow k > l)$$

Schechtman's proposal is that this formula states the existence of a measure that is greater than any finite number, though it is not itself a number.

It is hard to see how one can compare two heterogeneous things (l a number and k which is not a number) with respect to size as Schechtman proposes.[22] Let us waive this difficulty for now and ask: if it is not a number, then what is it? Schechtman tells us that it is a mode of quantity. On Schechtman's view, this mode of quantity is absolute space since space is a mode of quantity.[23] So we can take k as the mode of quantity identical to our idea of (absolute) space in the formula above – an idea which is empirically derived from an actually infinite absolute space. There's *prima facie* evidence for Schechtman's proposal in what Locke says in *An Essay Concerning Human Understanding* (*Essay*, henceforth) II.xvii.4:

[19] Schechtman (2019, fn. 17, p. 1123; fn. 26, p. 1127; and pp. 1140 - 1141)

[20] This is not the precise mathematical notion studied in measure theory. She uses this argument in order to disentangle Locke's *quantitative* notion of infinity from Leibniz's *iterative* notion of infinity and from Descartes' *ontic* notion of infinity. One of her most interesting claims is that while Locke accepts the notion of an infinite measure, Leibniz rejects both the notion of an infinite number *and* the notion of an infinite measure, although Leibniz accepts the *iterative* infinite.

[21] See §3.1 below.

[22] See Mancosu (1996, 35) where one of the properties of magnitudes in classical geometry is that only homogeneous magnitudes can be ordered by a total relation $<$.

[23] See Schechtman (2019, 1129 - 1130) for Schechtman's argument and Locke's *Essay* II.xvii.1 for the claim that the idea of space is a mode of quantity.

> It is a quite different consideration, to examine whether the mind has the idea of such a boundless space actually existing, since our ideas are not always proofs of the existence of things: But yet, since this comes here in our way, I suppose I may say, that we are apt to think that space in itself is actually boundless; to which imagination, the idea of space or expansion of itself naturally leads us.

Although Locke here appears to say that space is actually "boundless," I don't think that this is a counterexample to the view that it is common ground in the early modern period that there are no infinite lines or magnitudes. One reason is that in Schechtman's paper, there is no careful distinction between magnitudes and numbers. The discussion is weighted more heavily on the side of explicating quantities in terms of numbers or measures than in characterizing what magnitudes are and whether magnitudes can be infinite.[24] Furthermore, there is no discussion of what has come to be known as the *Eudoxus-Archimedes* Axiom (*The Elements* Book V. Definition 4) in Schechtman's paper; yet this axiom makes the connection between numbers, measures and magnitudes explicit and *rules out the existence of infinite magnitudes.*

Eudoxus-Archimedes Axiom
Magnitudes are said to have a ratio to one another which are capable, when multiplied, of exceeding one another.[25]

While Schechtman is right that numbers measure magnitudes, they do so in terms of the theory of proportions (Books V, VII and X in *The Elements*) and in accordance with this axiom.[26] Locke would have known that this axiom rules out the existence of infinite magnitudes regardless of his remarks regarding the infinity space.[27] Moreover, this axiom helps make sense of what Locke is saying in the chapter *On Infinity* (*Essay* II.xvii.7):[28]

[24] Schechtman (2019, 1120 - 1123)

[25] Cf. Mancosu (1996, 36) who puts it this way: Given any two magnitudes A and B such that $A < B$, there is a natural number n such that $nA > B$. This axioms rules out infinite magnitudes because if B is infinite and $A < B$, there is no natural number n such that $nA > B$. It also rules out infinitesmals, but this requires a more careful discussion. See Mwakima (2020) for a development of Berkeley's argument against infinitesmals.

[26] See §3.1 below for more discussion.

[27] See Mancosu (1996, 36) who speaks of the seventeenth century slogan, "There is no proportion between the finite and the infinite."

[28] Cf. Schechtman (2019, fn. 26, p. 1127). It is interesting that Berkeley makes use of the same distinction which he attributes to Locke in his short monograph *On Infinities* (W4: 234 - 239) read before the Dublin Philosophical Society on 19 November 1707. "Now I am of opinion that all disputes about infinites would cease, & the consideration of quantitys infinitely small no longer perplex Mathematicians, would they but joyn Metaphysics to their Mathematics, and condescend to learn from Mr. Locke what distinction there is betwixt infinity and infinite." W4:239

> Therefore I think it is not an insignificant subtilty, if I say that we are carefully to distinguish between the idea of the infinity of space, and the idea of a space infinite: The first is nothing but a supposed endless progression of the mind, over what repeated ideas of space it pleases; but to have actually in the mind the idea of a space infinite, is to suppose the mind already passed over, and actually to have a view of all those repeated ideas of space, which an endless repetition can never totally represent to it; which carries in it a plain contradiction.

Here Locke is cautioning us against identifying the truth regarding the infinity of space (what he called the boundlessness of space) with the assertion that space is infinite. One way of understanding his point here is that asserting that space is infinite (in magnitude) would contradict the Eudoxus-Archimedes axiom because to represent an infinite magnitude, we'd need to take an infinite number of repetitions of the common measure. Since an *actual* infinite number of repetitions would violate the Archimedian axiom, it follows that we can't represent an infinite magnitude or space. Thus, Schechtman's proposal does not challenge the assumption that it is common ground that there are no infinite magnitudes or lines.

The other way taken by scholars of responding to Berkeley's argument is to point out that Berkeley is missing the obvious property of convergent geometric series.[29] A convergent geometric sequence is an infinite sequence s_n with $n \in \mathbb{N}$ of terms with a common ratio $|\frac{s_{n+1}}{s_n}| = |r| < 1$ between successive terms (s_n and s_{n+1}). An infinite geometric series is an infinite series such that the sequence of partial sums S_n converges. For example, the sum of the terms in the geometric sequence $\langle 1, \frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2^n}, \ldots \rangle$ for $n \in \mathbb{N}$ is 2. This sequence converges since the common ratio $|r| = \frac{1}{2} < 1$.

But I don't think that Berkeley is ignorant of the existence of convergent geometric series or denying the theorems which support them. First of all, *The Elements* does not discuss the notions of convergence and divergence of infinite series. So pointing out the existence of convergent sequences, supports rather than refutes Berkeley's argument that infinite divisibility is not in *The Elements*. The mathematics of infinite series was made precise in the 19th century with the work of Cauchy and Weierstrass aimed at rigorously reformulating analysis.[30]

[29] This is a point that Fogelin (1988, 52 - 53) and Franklin (1994) make.

[30] The terms 'convergent' and 'divergent' were used by James Gregory in 1668 but he did not develop the ideas. Newton only affirmed that power series converge for small values of the variable and for the geometric series. Leibniz showed that series whose terms alternate in sign and decrease in absolute value monotonically to zero converge. See Kline (1972, Vol. 2, 461) and for Cauchy and Weierstrass see Kline (1972, Vol. 3, 948, 952, 963ff.).

Moreover, according to commentators, it is Aristotle, not *The Elements*, who was one of the first to use "convergence of a geometric sequence" synonymously with "potentially infinitely divisible" as a response to Zeno's paradoxes.[31] Zeno in supposing that magnitudes are infinitely divisible, *intended this to imply that the magnitudes are actually divided into infinitely many parts.* He did not intend it in the restricted Aristotelian sense of merely potentially infinitely divisible. His paradoxes were that a *supertask* would have to be completed. A supertask is a task involving actually infinitely many steps completed in finite time.[32] For example, in the Dichotomy Paradox, the motion can never begin because to start from the beginning of an interval to the half-way point of the interval, one would have to traverse an actually infinite number of monotonically decreasing intervals of space approaching the beginning of the interval. So if someone merely pointed out to him that a geometric series converges as Aristotle did – speaking in terms of potential infinite divisibility – Zeno would have been unconvinced that this solves the Dichotomy Paradox. In other words, Aristotle's shows *why* motion is possible (namely that the sum of a convergent series is finite) not *how* it is possible (how can actually infinite many steps be completed in a finite time). Here we see one way how Aristotle's *philosophical* thesis of potential infinite divisibility became associated with a *mathematical theorem* that asserts the existence of convergent geometric sequences. The other way has to do with Aristotle's reconceptualization of continuity which I discuss below.

The Port Royal Logicians (Antoine Arnauld and Pierre Nicole), Isaac Barrow and John Keill[33] continued this Aristotelian thought in the late seventeenth century and early eighteenth century, arguing that the convergence of geometric series counts as a reason in favor of infinite divisibility. Berkeley clearly read these mathematicians' work as evidenced by his notebook entries.[34] It is within this intellectual milieu that Berkeley enters in order to restore order. The intellectual (albeit virtual) exchange between Barrow and Keill on the one side supporting infinite divisibility in mathematics and Berkeley on the other denying it has been discussed in detail in Jesseph (1993, 63 - 67). However, one argument that Jesseph does not discuss is the argument based on the convergence of geometric series. Barrow (1734, 157), in *Lecture IX*, puts it this way:

> [I]t is plainly taught and demonstrated by Arithmeticians, that an infinite series

[31]See Heath (1956, Vol. 1, 233 - 234) for discussion. For the reference in Aristotle see *Physics* 206b4-206b12.

[32]For the supertasks reading of Zeno's paradoxes see Black (1967) and Manchak and Roberts (2016).

[33]Keill became the Savilian Professor of Astronomy in Oxford in 1712. For Keill's view I have used Keill (1745) *An Introduction to Natural Philosophy: or Philosophical Lectures Read in the University of Oxford 1700 A.D.*

[34]See PC 263 W1:33 for example.

> of fractions, decreasing in a certain proportion, is equal to a certain number; e.g. that such a series of fractions decreasing in a *subsesquialter* proportion is equal to two, in a *subduple* proportion to unity, in *subtriple* to one half; from whence it is not inconsistent for something finite to contain in it an infinity of parts.

Since Berkeley read these authors, it is not true that he was ignorant of the possibility that convergent geometric series would be counterexamples to his view that any magnitude that contains infinitely many parts must be infinite.

2.2 What are parts?

Why, then, would Berkeley have been convinced that his arguments were sound? In the previous subsection I rejected the view that Berkeley was simply ignorant of convergent geometric series (a series with infinitely many parts yet finite in size). So in order to see the soundness of Berkeley's argument, we need to look more carefully at the meaning of the term 'extension', the distinction between parts and wholes, and what infinite divisibility (or its denial by Berkeley) even means.

Many have supposed that Berkeley's denial of infinite divisibility entails that for every line L there is a finite number of divisions n that can be done on L such that for all $m > n$, L is not divisible further. This reading is wrong because Berkeley doesn't think there are indivisible lines. His geometrical minima are points not lines.[35] At the same time, his denial of infinite divisibility doesn't entail that there are a finite number of points in a line. The notion of cardinality which we get later with Cantor – such that a line has actually infinitely many points – is alien to him. Berkeley consistently uses the notion of parts not points in his arguments against infinite divisibility. This means that Berkeley is assuming an actual ordered dense point set conception of a line and is analyzing the divisibility of lines in *metrical* terms not in terms of *cardinality.* This metrical approach is justified by Postulate 1 (To draw a straight line from any point to any point) and Postulate 3 (To draw a circle with any centre and distance). This in turn means that when Berkeley says lines are finitely divisible, he means that for any two points x, y in a finite line such that $x < y$, the Euclidean distance $d(x, y) = y - x$ or parts[36] between these two points is some finite *number* of *units of distance.* His argument in PHK §124 is that the sum of this finite number of units of distance in a finite line can only be finite. Suppose otherwise, then there

[35] See especially the *New Theory of Vision* §§54 - 61; PHK §127 and *De Motu* §15 at least.

[36] Parts here is in the plural sense. See below for more discussion of what parts in the plural sense means in Euclid.

is an infinite *number* of units of distance between a point a, b in a finite line. But this is absurd unless the line is infinite. For example, take the points to be the boundaries of the finite line contained within the interval $[0, 5]$. Today we know that there are infinitely many points in this interval. But once one adopts the Euclidean distance, then there are 5 units of distance. Thus, by denying infinite divisibility, Berkeley wants us to draw at least two conclusions. First, the sum of the $d(x, y) = y - x$ units of distance between all points x, y in a finite line is finite. That is, it is infinite only if the line is infinite. Secondly, Berkeley denial of infinite divisibility presupposes that lines can be composed of indivisible points or geometrical minima since Aristotle's thesis regarding infinite divisibility is equivalent to the thesis that continuous quantities are fundamentally non-atomic.

> Nothing that is continuous can be composed of indivisibles: e.g. a line cannot be composed of points, the line being continuous and the point indivisible [...] [I]t is plain that everything continuous is divisible into divisibles that are always divisible; for if it were divisible into indivisibles, we should have an indivisible in contact with an indivisible, since the extremities of things that are continuous with one another are one and are in contact. The same reasoning applies equally to magnitude, to time, and to motion: either all of these are composed of indivisibles and are divisible into indivisibles, or none [of these are]. If time is continuous, magnitude is continuous also [...] If time is infinite in respect of divisibility, length is also infinite in respect of divisibility.
>
> Aristotle, *Physics* Book VI 231a18 - 20; 231b16-232a17; 233a13-233a21

Aristotle's thesis of infinite divisibility applied to magnitudes (or lines) is equivalent to the thesis that lines cannot be composed of points. By denying infinite divisibility, Berkeley is rejecting this Aristotelian view in favor of the view that the division of a line can't go on infinitely because eventually *contra* Aristotle we must arrive at mathematical atoms or points. So, the real question is this: do *The Elements*'s starting assumptions allow us to draw the conclusion that finite lines are infinitely divisible or must division eventually terminate into points? This is the issue and the reason why the convergence of geometric series is not a counterexample to Berkeley's denial of infinite divisibility in Aristotle's sense. Today, with a point-set conception of the continuum, the convergence of an infinite sequence is proved by showing that after a *finite* – not infinite – large number N, all subsequent terms s_m with $m > N$ are so close to each other that they are virtually indistinguishable i.e. the distance between them is almost negligible or there is "no part" between them. So beyond N, the Euclidean distance $d(s_m, s_{m+1}) = s_{m+1} - s_m$ between any two terms adds nothing significant to the already total finite distance of the terms before N. This is why a geometric series converges to a finite number. It is not a counterexample to Berkeley's sense of denying infinite divisibility. What we need to find out is whether the picture I am painting is consistent with *The Elements* as Berkeley or Barrow would have read it and whether Berkeley is right that infinite divisibility does not follow from anything in *The Elements*. The answer depends on what *The Elements* meant by parts and points.

The talk of "no part" in the context of convergent geometric series ought to remind us of the starting point (pun intended) of *The Elements*. Famously, *The Elements* begins with the definition of a geometrical point (points, henceforth) as "that which has no parts." But what does having "no parts" mean? One possibility could to be use our observation in the case of convergent infinite sequences and say that "having no parts" means "having no internal distance", or what the early modern philosophers, following Descartes[37] and the Port Royal Logicians[38], express in terms of "lacking extension." *The Elements* is not clear on this. Kline (1972, Vol. 3, 1008) notes that one criticism that Moritz Pasch (1843 - 1930) made of *The Elements* had to do with *The Elements*'s definitions of 'point' and 'part.' In the geometrical Book V of *The Elements*, we read, "A magnitude is a part of a magnitude, the less of the greater, when it measures the greater." Taking this geometrical characterization together with what we find in the arithmetical books (Books VII, VIII, IX of *The Elements*), we may say that a part of a magnitude or number is what we call today a *factor* or *integral divisor* according to some unit of distance or measure. Part*s* (plural) are what we call today a *fraction*, although ancient geometers did not make use of expressions which we use today when we talk about fractions.[39] The introduction of fractions as denoting quantities or real numbers had to wait until 1500 or so.[40] The other mention *The Elements* makes of 'parts' is in the discussion of the *Common Notions*. Common Notions were self-evident truths with such widespread acceptance that most people adopted them without proof. Some of these common notions were:

1. Things which are equal to the same thing are also equal to one another.

5. The whole is greater than the part.

[37]In *Rules for the Direction of the Mind* Rule XIV it is noteworthy that Descartes disentangles his notion of 'extension' from 'quantity.' In Meditation V, quantity is only applied to continuous quantity and Descartes speaks of extended quantity. It is a thorny issue to try to understand what extension is for Descartes so I will not get into that here. See Garber (1992) especially Chapter 3 and 5.

[38]See Arnauld, Antoine and Nicole, Pierre (1996, 231-232) where the words "zero extension" are used.

[39]Cf. Heath (1956, Vol. 2, 115) for Heath's discussion of parts of a magnitude and Heath (1981, Vol. 1, 42) for a discussion on fractions.

[40]See Kline (1972, Vol. 1, 251).

Despite this widespread belief in antiquity, today it is hard to accept the fifth common notion. Today we have set theory which offers us the resources to distinguish between the *membership relation* $\in$ that holds between an element (or individual) and the class of which it is a member; and the *subset relation* $\subset$ that holds between a subclass and the class of which it is a part. Thus one may be tempted to think of the part-to-whole relation in *The Elements* in terms of the subset relation and argue that the common notion *The Elements* is presupposing is open to obvious counterexamples. We know, for example, that in the case of infinite sets, the whole is not necessarily greater than the part. While this result would have been paradoxical to Galileo, Leibniz and Berkeley, it would be anachronistic to try to refute their view on the basis of modern developments in mathematics.[41]

We must seek to understand Berkeley and his contemporaries on their own terms and resist thinking about the part-to-whole relation in terms of the subset relation. Specifically, we must resist the temptation of thinking, as we do today, that points are the elements in a line (i.e. that the geometric continuum is a set of actually infinite points) while lines are subsets of other lines. The point-set conception of the geometric continuum is a modern invention beginning most explicitly with Cantor and Dedekind.

What *The Elements* means by the part-to-whole relation involves homogeneity, the property of two or more things being similar in some respect. We may say that A is a part of a whole B iff A is homogeneous with B but not equal to (i.e. strictly less than) B.[42] In *The Elements* and most ancient geometers, the part-to-whole relation in geometry is a relation between homogeneous *quantities* since it is only the category of quantity, according to Aristotle, that admits of the relation equal-to, less-than or greater-than.[43] For example, the parts of a (whole) line will be other (homogeneous) lines. A part of a (whole) multitude, such as a collection of coins, will be another smaller collection of coins. This is all *The Elements* intended by this common notion.

This suggests the following interpretation of *The Elements*'s definition of a point. In

[41] I revisit Galileo's paradoxes (Aristotle's wheel and the equinumerosity of the set of square numbers with the natural numbers) and Leibniz's paradox of infinite number and their relation to Berkeley's view on the infinite in Mwakima (2020) where I discuss the distinction between distance and cardinality. There, I argue that Berkeley is construing spatial magnitudes in terms of their metrical and order properties not in terms of the cardinality of points in them. There's a strong parallel to Berkeley's thinking in Bolzano. Bolzano realized that there is a one-to-one correspondence between the real numbers in the interval $[0, 5]$ and the interval $[0, 12]$ given by $y = \frac{12x}{5}$ a function from $[0, 5]$ to $[0, 12]$ but was still reluctant to accept that these two intervals have the same "size." We can explain Bolzano and Berkeley's puzzlement because they were thinking about size in distance or metrical terms rather than in terms of cardinality.

[42] See *The Elements* Book V and Book VII also.

[43] "[The] most distinctive of a quantity would be its being called both equal and unequal." *Categories* 6a26-6a36.

saying that point has no parts, *The Elements* means that there is nothing strictly less than a point which is homogeneous to it. One point to take away here is that *The Elements*'s definition of a point does not *by itself* tell us anything about the divisibility (or lack thereof) of points. The definition clearly does not mention divisibility. Secondly, *The Elements*'s definition does not deny that a point has what Plato called *onkos* (roughly, size or volume; more of this below). I will take this to mean that *The Elements*'s definition of a point does not deny that a point has *minimal size*.[44] So saying that there is nothing less than a point which is homogeneous to it does not entail that a point is nothing (i.e. that it has no size) as Hume famously thought in considering alternatives to his position. Hume considered his position (there are minima with color and solidity) to be the middle ground between infinite divisibility and mathematical points.[45] What it does entail is that there is nothing *smaller* than a point which is homogeneous to it. This will be important in understanding Berkeley's mathematical atomism and his doctrine of *minima tangibilia*.

So where did this pervasive characterization of points as being indivisible originate? I claim that this identification of a point with the indivisible started with Aristotle's idiosyncratic conception of mathematics. This conception was developed in order to refute the physical atomists (there are indivisible physical atoms that compose matter). In doing so, Aristotle conflated the physical atomist thesis with what I have called the mathematical atomist thesis (magnitudes are composed of actual points with distance and order relations between them). The sixth century CE Neoplatonist Simplicius, one of the few extant ancient commentators on Aristotle's *Physics*, has this to say in his commentary on Aristotle's *Physics* Book VI (this is the book that deals with continuity):

> Aristotle set up the logical division of the divisible into either indivisibles or forever divisibles, so that he might comprise the continuous in that which is divisible into forever divisibles. Simplicius (2014, 23f) Trans. lines 931, 5 - 10 in MSS.

More recently, Miller, Jr. (1982, 88) has written,

> Aristotle reformulated the old difficulties in his own terms and defined concepts in order to resolve them...He presents his own theory of the continuum as the

[44]There are complications with my attempt to reconstruct what Euclid might have meant by 'point.' These complications arise in view of recent developments in measure theory where a point is, indeed, assigned measure 0. I revisit such complications in Mwakima (2020). See Skyrms (1983) for an excellent introduction to the basics of measure theory.

[45]See *A Treatise on Human Nature* II.iv. The literature on Hume and infinite divisibility is vast. Good places to start are Jacquette (1996), Pressman (1997) and Holden (2002).

> only way out of an ancient dilemma which seeks to show the absurdity of continuous magnitudes.

I return to the dilemma in a moment. The important take away, for now, from Simplicius and Miller, Jr. is that Aristotle was reconceptualizing the debate and that this involved identifying the continuous with the infinitely divisible. The identification of the point with the indivisible is clearly stated in Aristotle's *Metaphysics* V.6 1016b18-1016b30. Here he writes:

> But everywhere the one is indivisible either in quantity or in kind. That which is indivisible in quantity and qua quantity is called a unit if it is not divisible in any dimension and is without position, a point if it is not divisible in any dimension and has position.

It is not clear who Aristotle's sources were for this characterization of points and units.[46] What we do get clearly from Aristotle is one way to conceive of mathematical points is that they are indivisible. But if they are indivisible, does it follow that the points have no *onkos* or are nothing? '*onkos*' is a technical term used in different contexts – some of these contexts are theatrical. Ancient scholars[47] vary in translating '*onkos*' as volume, measure or simply spatial extension and are divided on this question and what implications it has for our conception of points with respect to divisibility.[48] For our purposes, it is sufficient to note that it is the translation of *onkos* as spatial extension that has survived up to the early modern period when Descartes and the Port Royal Logicians began speaking of a point as that which has no (spatial) extension.

Laying aside the difficulty of how to translate or understand *onkos*, this question raises a dilemma. On the one hand, if someone says that the mathematical points have no size or spatial extension, then they are "nothing" and cannot be parts of magnitudes. For the parts of magnitude are other (homogeneous) magnitudes with size. On the other hand, if one says that the mathematical points have size (i.e. they are proper parts of magnitudes), then they are not indivisible after all. Zeno, as presented by Aristotle, exploited this dilemma with relish using his paradoxes. On the one hand, he forced Aristotle to reject

[46] Cf. Proclus (1970, 78) Trans. lines 95.21 - 96.14. "A point is a unit that has position." The Pythagorean definition does not mention divisibility.

[47] See (Pfeiffer, 2018, 130 - 131) for example.

[48] See Vlastos and Owen discussed in Furley (1967, 67). Furley notes that Vlastos writes of Zeno's assumption, "that anything which does have size is at least logically divisible and has at least logically discriminable parts." But he also mentions Owen who writes that Zeno assumes without argument that the conjunction of size with theoretical indivisibility would be a contradiction.

indivisible magnitudes in favor of infinite divisibility. On the other hand, the Epicureans and ancient atomists exploited the assumption that a magnitude is infinitely divisible into parts with size to argue that this would imply that the original magnitude is infinite in size. So they accepted indivisibles.[49] We've already met this Epicurean argument in connection with Berkeley.

As mentioned, Aristotle got himself out of this dilemma by arguing that the mathematical points have no size. Faced with the conclusion that they cannot be parts of magnitudes or are "nothing", he argues that points exist potentially. That is, rather than accept that points are non-entities (since they lack *onkos*), Aristotle opted to say that a point is actualized whenever a magnitude is split, say into two smaller magnitudes. Here's how Miller, Jr. (1982, 98) puts it:

> Aristotle refutes the nihilistic horn [the name Miller, Jr. gives for the first horn of the dilemma we've been discussing], used by atomists, by showing that even though division is possible and a point exists everywhere in the potential mode, it does not follow that magnitude reduces to points. For the existence of every actually existing point is conditional upon the existence of two segments with magnitude into which the subsection is divided.

Thus the same point is the limit or extremity of the two magnitudes resulting from the split. That is, the point existed potentially before the split but now exists actually as a limit or extremity of the two separate lines after the split.

How does Aristotle know that there is no location on a magnitude (such as a line) where there is a "gap" that would prevent the splitting of a magnitude at that location? He does not know this. Aristotle has to either prove that a magnitude (say a line) is continuous in either the dense or Cantor-Dedekind complete sense first; or assume that it is before he can argue that points exist potentially. In fact, Aristotle neither assumed nor proved any of these alternatives since for him a line was not composed of points. What Aristotle did is to assume that you can always bisect a line segment into two equal segments. Continuity for him consisted in the identity of the right limit of the left segment and the left limit of the right segment.[50] On the basis of this analysis of the existence of points and continuity, Aristotle drew the conclusion that continuous magnitudes (such as lines) are infinitely divisible on the basis of the claim that bisections can be done an indefinite number of times.[51] Some geometers (see the quotation from Proclus in the next section), following

[49] See Diogenes Laertius (2018, 507 - 522) for Epicurus' *Letter to Herodotus*. I develop the connection between Epicureanism and Berkeley's mathematical atomism in Mwakima (2020).

[50] I thank Brian Skyrms for help in clarifying this part of my paper.

[51] See *Physics* 207b10.

Aristotle, then understood the infinite divisibility of finite lines to be a consequence or assumption of what is now *The Elements* Book I Proposition 10. How warranted were geometers to draw this consequence or make this assumption? Let us look at this next.

2.3 Infinite Divisibility and *The Elements* Book I Proposition 10

Proposition 10 in *The Elements* Book I is the proposition 'To bisect a given straight line.' The proof is familiar to most people from elementary geometry using compass and straightedge. The important point is that if one analyzes the proof, *The Elements* does not draw the conclusion that this process can be iterated infinitely many times. We know that Aristotle predated Euclid's textbook *The Elements* and that Aristotle and his students at The Lyceum had a different geometry textbook that according to historians (Heath, 1981, Vol. 1, 321) was authored by Theudius. There was also an arithmetical textbook *Elements of Arithmetic* apparently authored by Archytas (430 - 365 BCE) who also predates Aristotle.[52] We may never know how Theudius proved this theorem and what conclusion he drew because that textbook is lost. Thus, it is impossible to know definitively whether Euclid and Aristotle differed in their conception of ancient geometrical practice. Recent scholars Linnebo and Shapiro (2019, 164) speculate:

> Because of the structure of the geometric magnitudes (to echo Lear (1982)), we have procedures that can be iterated indefinitely, and we speak about what those procedures could produce, or what they will eventually produce if carried sufficiently (but only finitely) far. In holding that these geometric procedures can be iterated indefinitely, Aristotle again follows the mathematical practice of the time, this time in opposition to his other major opponents, the atomists, who postulate a limit to, say, bisection.

Notice from Linnebo and Shapiro that this procedure is only indefinitely but finitely carried out. For Aristotle, the 'infinite' in 'infinitely divisible' is really just large finite N for N as large as we wish. So it was a little misleading for Aristotle to claim that *infinite* divisibility follows from the bisection theorem.

Reiterating this point, this is how Proclus (1970, 216 - 217, Trans. lines 278 - 279 in MSS.) puts it in his commentary on Book I, Proposition 10 (my emphasis).

> [If a line] is not composed of indivisible parts, it will be divisible to infinity. This, they say, appears to be an agreed principle in geometry, that a magnitude

[52]See Heath (1956, Vol. 2, p. 295).

> consists of parts infinitely divisible. To this we shall give the reply of Geminus, that geometers do assume, in accordance with a common notion, that what is continuous is divisible. The continuous, we say, is what consists of parts that are in contact, and this can always be divided. *But they do not assume that what is continuous is also divisible to infinity*...it is an axiom that every continuum is divisible; hence a finite line, being continuous, is divisible. This is the notion that the author of the *Elements* uses in bisecting the finite straight line, not the assumption that it is divisible to infinity. That something is divisible and that it is divisible to infinity are not the same.

Proclus is urging us to distinguish what we are in fact bisecting. After the $n > 1$ bisection stage, we are strictly speaking not bisecting the original line. So it is false to say that bisections of the original line can be done infinitely many times. The theorem says that for each line the bisection can be done only once since all that continuity guarantees is that any line segment is divisible i.e. there are no indivisible lines. But he also points out that using *The Elements* Book I Proposition 10 as a proof for infinite divisibility of the original line is an extrapolation or fallacy. Aristotle was one of those people who fallaciously made the extrapolation from the bisection or divisibility of *each* line (true) to potential infinite divisibility or bisection of the same original line (false). While it is true that one needs to assume continuity as a common notion to argue for the actual existence of a point as the limit of the two resulting line segments from bisection at a potentially existing midpoint; Aristotle, eager to refute the physical atomists, was caught in a fallacy of confusing the claim that *every line* is bisectable *once* (true) with the claim that the *same line* is bisectable at every stage $n > 1$ (false). This is another source of evidence that Berkeley is right to point out that (potential) infinite divisibility is not a theorem or axiom in *The Elements*.

However, there is the issue of incommensurable magnitudes. Many have taken this to be evidence for infinite divisibility. In fact, in this same commentary on Proposition 10, Proclus (1970, 217) says that infinite divisibility follows from the existence of incommensurable magnitudes. Later philosophers such as the Port Royal Logicians[53] took the existence of incommensurable magnitudes to be the definitive demonstration that there are no indivisible parts in magnitudes. Incommensurability poses a threat for anyone who denies infinite divisibility (like Berkeley) only if such a person: (1) believes that there are indivisible lines; and (2) believes that the number of indivisible lines that a line can be divided into corresponds to its size. For if (1) and (2) are true, then suppose that the

[53] Arnauld, Antoine and Nicole, Pierre (1996, 231)

hypotenuse of a right triangle with side of unit length can only be divided into a finite number of lines m and the side can only be divided into a finite number of lines n where $m > n$ and m and n are in their least terms (i.e. having a greatest common divisor of one). Then the existence of the ratio $m : n$ would contradict the well-known theorem that there are no numbers m, n in their least terms such that the proportion $m : n :: \sqrt{2} : 1$ holds.[54]

But earlier I showed that Berkeley denies (1) because Berkeley doesn't believe there are indivisible lines. So incommensurability doesn't pose a threat for Berkeley if he denies infinite divisibility. But does incommensurability really imply or presuppose that magnitudes are infinitely divisible? Recall that in the quotation that gave me the clue for how to approach this paper, Berkeley thought, *contra* Barrow, that deducing infinite divisibility from the existence of incommensurables is a *petitio principii* in the passage we began with. Why did he think so? This issue needs to be investigated even though I have shown that the arguments in favor of infinite divisibility from incommensurability doesn't threaten Berkeley's denial of infinite divisibility. Let me now turn to assessing the issue of incommensurables and whether they presuppose or imply the infinite divisibility of continuous quantities. Here, I will show that the historical association of infinite divisibility and incommensurability arises from different conceptions of mathematics (the Aristotelian and the Pythagorean) and a historical confusion regarding the upshot of the number theoretic process that led to the discovery of incommensurable magnitudes by the Pythagoreans in the first place. Berkeley's suggestion is that incommensurability (which arises in number theory) needs to be kept distinct from infinite divisibility (which arises in geometry). This is one instance where Berkeley, unlike Barrow, is insisting on the distinction between geometry and arithmetic (or number theory).

3 Aristotelian and Pythagorean views of Mathematics

3.1 Aristotle on Quantity

Aristotle's views on quantity in his collected works begin with the account of quantity in the *Categories* and is developed through the *Physics* and the *Metaphysics*. Throughout these accounts, Aristotle consistently distinguishes between discrete quantities *arithmos* (number) and continuous quantities *megethos* (magnitude). The genus term 'quantity' is the Greek word '*poson*.' But there's also the question of how to translate terms like *to pelikos* (how great), *onkos* or extension/volume and *metron* or measure as ways of discussing

[54] I thank Jeremy Heis for a fruitful discussion about the issue of incommensurability and how it bears on infinite divisibility. See §3.3 below for discussion of this proof.

quantity. The taxonomy is complicated and opens up a lot of philosophical debate.[55] What is important for my purposes is that however this taxonomy ends up being sorted out, it is only one of the many other possible conceptions of mathematics that were available during Aristotle's time. At the heart of Aristotle's philosophical defense of infinite divisibility and the potential existence of points, I will argue, is that he held a different conception of mathematics. In doing so, he betrays an unfamiliarity with the import of the Pythagorean discoveries in mathematics; and the subsequent codification of these discoveries by Eudoxus in the theory of proportions in Book V and some of Theaetetus's discoveries that ended up being codified in Book X of *The Elements*. To be sure, Eudoxus and Aristotle were contemporaries and Theaetetus predated both of them. We may never know whether Aristotle was acquainted with Eudoxus's discoveries on the theory of proportions or whether Theaetetus's contribution, which we find in *The Elements* Book X was included in the Theudius geometry textbook that was used in The Lyceum. In what follows (§3.2), Aristotle's remarks in the *Metaphysics* suggest an unfamiliarity with how to place Pythagorean number theoretic discoveries on rigorous geometrical foundations.

3.2 Pythagorean Mathematics: A non-Aristotelian Conception of Mathematics

It is difficult to assess what Pythagoras actually believed because there is no extant work written by Pythagoras. Any attempt to reconstruct what subsequent Pythagoreans actually believed cannot therefore be substantiated by anything from Pythagoras himself. To get a sense of the Pythagorean view of mathematics we have to rely on second hand accounts from philosophers like Plato and Aristotle some of whom, unfortunately, had an axe to grind; and commentators like Iambilichus, Proclus and Diogenes Laertius. In *Metaphysics* 985b23-986a13, Aristotle, for example, writes:

> Contemporaneously with these philosophers and before them, the Pythagoreans, as they are called, devoted themselves to mathematics; they were the first to advance this study, and having been brought up in it they thought its principles were the principles of all things.

The history of arithmetic begins in Greece with Pythagoras who is believed to have lived during the sixth century BCE. Historians speculate that Pythagoras was led to his number-monism (all there is are numbers and proportions between numbers) by his discovery in music theory of the *harmonical* proportion. That is, the fifth and the octave of a note

[55] See Pfeiffer (2018) for the most up to date philosophical discussion of this taxonomy.

could be produced on the same string by stopping at $\frac{2}{3}$ and $\frac{1}{2}$ of its length, respectively. Gow (1968, 68) writes about how led by such considerations,

> Pythagoras considered number to be the basis of creation: he looked to arithmetic for his definitions of all abstract terms and his explanation of all natural laws.

Thus, beginning with number-monism, Pythagoreans went on to develop number theory by classifying numbers as: odd, even, square, cube, triangular, perfect, defective, amicable etc. Proportions were either arithmetical, geometrical and harmonical.[56]

Given this Pythagorean number-monism, the first distinction we can make between the Aristotelian conception of mathematics and the Pythagorean is that on the one hand, for the Pythagoreans there are no species of quantity. Aristotle is aware of this, writing in *Metaphysics* 1080b17-1080b21:

> Now the Pythagoreans, also, believe in one kind of number – the mathematical; only they say it is not separate but sensible substances are formed out of it.

On the other hand, for Aristotle, magnitude and numbers are both species of the genus quantity. The differentia, therefore, had to be sought. This difference was, for Aristotle, in terms of continuity and discreteness. Aristotle goes to great extent to defend his view of quantity first in the *Categories* and more fully in the *Physics*. In the *Physics*, he introduces subtle distinctions between whole and part; and between things being successive (next to each other), contiguous (touching), and finally continuous (*synechi* syn = together; echo = to have/hold) which in the Latin was translated *contenere* (con = together; tenere = hold). So the continuous is that which is "held-together." The depth and rigor of Aristotle's penetrating analysis going from weaker to stronger conditions for what is required for continuity is in an extended discussion in *Physics* beginning in Book III all the way to Book VIII. Along the way, the association of infinity with continuity is made – an association that is with us to this very day. Further, Zeno's paradoxes of motion are considered and supposedly rebutted using the machinery developed until that point.[57]

One key difference between Aristotle and the Pythagoreans in this regard, is that for the Pythagoreans only numbers (i.e. positive integers greater than 1) can be answers to the question of quantity (*poson*). These are questions that take the form "How many (much) X?" (*poson*) or the form "How great is X?" or "What size is X?" (*to pelikos*). Here is the

[56]For details and historical references see Heath (1981, Vol. 1, 72 - 84) and Proclus (1970, 52 - 57).

[57]This is not the place to undertake a detailed analysis of Aristotle's analysis of continuity. For a good discussion see Miller, Jr. (1982) and Sorabji (1982). For a more recent discussion see Pfeiffer (2018).

important point, which Aristotle shows an unfamiliarity with. In the case of *magnitude*, numbers answer the question “How great?” or “What size is X?” in terms of *proportion* between *two* numbers.[58] Furley (1967, 52) writes:

> The Pythagorean method relied on finding proportions, and not on counting atomic constituents. It is the proportion 2:1 which constitutes the octave, no matter what the units may be.

Just as in the case of harmonics, the Pythagorean answer to the magnitude question “How great is X?” or “What size is X?” in geometry is a ratio or proportion (a proportion is an equality between ratios) involving *numbers* determined by *measuring* the two magnitudes with respect to size. Heath (1981, Vol. 1, 153) speculates that the Pythagorean theory of proportions was only applicable to commensurable magnitudes and that it was Eudoxus’s work (which we find in Book V of *The Elements*) that generalized this theory to include incommensurables. Thus, unlike Aristotle, who sought to distinguish *arithmos* from *megethos*; for the Pythagoreans, there was only *arithmos* which was used to understand the *megethos*.

Let me put this in another way. The Pythagoreans started with number theory. Numbers were understood, for example, as even or odd; perfect; prime and so on. Corroborating Heath’s claims, Van Der Waerden[59]speculates that it was Eudoxus’s contribution that found its way to *The Elements* in Book V; and Theaetetus’s contribution that found its way to Book X, that sought to place Pythagorean number theory (or arithmetic) on rigorous foundations (geometry). Eudoxus’s and Theaetetus’s genius made it possible to embed the Pythagorean number theory into geometry using the general theory of proportions applicable to commensurable and incommensurable magnitudes. Euclid assembled these results in Book V and Book X respectively. The result is that on the Pythagorean conception of mathematics there was no need to have different answers to questions involving quantity (“How many (much)?”, “How great is X?” or “What size is X?”) in terms of discrete quantities and continuous quantities, as Aristotle thought. Rather, the answers are all in terms of numbers: positive integers or whole numbers in the case of “How many?”; or a ratio between two whole numbers in the case of “How great is X?” or “What size is X?”(magnitude). This is how Proclus (1970, 49 Trans. lines 61f) puts it:

> The theory of commensurable magnitudes is developed primarily by arithmetic and then by geometry in imitation of it. This is why both sciences define

[58] Proclus (1970, 53) credits Pythagoras for discovering the doctrine of proportions.
[59] See Van Der Waerden (1961, 107 - 126; 141 - 146; 165 - 168; and 175 - 179).

> commensurable magnitudes as those which have to one another the ratio of a number to a number, and this implies that commensurability exists primarily in numbers.

We now can see why the discovery of incommensurable magnitudes (i.e. magnitudes that cannot be expressed (*irrational or alogos*) as a ratio between two integers one of which is their greatest common divisor or unit) was such an astonishing discovery. The astonishment was not, as is often suggested, that there were "gaps" in the rational numbers that had to be filled or completed by irrational numbers in order to get the real number continuum. The astonishment is that the Pythagorean number-monism was being threatened.[60]

We all know that the first discovery of incommensurability was of what we denote today by '$\sqrt{2}$.' The Pythagoreans would have used their number theory to say that there are no two whole numbers m, n such that $m : n :: \sqrt{2} : 1$. In other words, $\sqrt{2}$ is incommensurable using 1 as the unit of measure. The proof is number theoretic since it is in terms of the distinction between odd and even *numbers*. Aristotle is clearly aware of this proof since he mentions it in *Prior Analytics* 41a26 - 27. But even though $\sqrt{2}$ was incommensurable, the Pythagoreans still had a way of expressing it in terms of a proportion between *known* magnitudes as follows: $\sqrt{2} : 1 ::$ **diagonal of right-isosceles triangle with side of length 1: one of the sides of the right-isosceles triangle**.

So, the Pythagoreans did not conclude that the rational numbers are incomplete ("gappy" or discontinuous) as we often hear. The Pythagoreans were not even thinking about these problems in terms of continuity or discontinuity at all. This can explain why *The Elements* is silent about its continuity assumptions except for Postulate 2 (To produce a finite straight line continuously in a straight line). The reason is that *The Elements* could never have doubted that magnitudes (such as lines) are continuous. We've already seen evidence from the commentary of Proclus that continuity was a common notion. But what the existence of incommensurables did do, was to motivate a program in search of a rigorous theory of proportions between magnitudes in order to study, classify and ultimately understand what those newly discovered incommensurables were. This was the theory that was developed by the magisterial Eudoxus and Theaetetus and immortalized in *The Elements*'s Book V and Book X.

This brings us to the second difference between Aristotle and the Pythagoreans. Because Aristotle makes the distinction between continuous and discrete, he holds that there are indivisible units in discrete quantities (number) but not in continuous quantities (magnitude). Consequently he mistakenly attributes to the Pythagoreans the view that there

[60] Cf. Heath (1981, Vol. 1, 155).

are indivisible magnitudes. That is, that the Pythagorean units (or indivisibles) have spatial magnitude. He writes:

> For [Pythagoreans] construct the whole universe out of numbers only – not numbers consisting of abstract units; they suppose the units to have spatial magnitude. But how the first unit was constructed so as to have magnitude, they seem unable to say. *Metaphysics* 1080b17-1080b21

What evidence does Aristotle have to assert the last claim? His claim is justified only because he held a different conception of mathematics from the Pythagoreans. Not only this, he also adds:

> The doctrine of the Pythagoreans in one way affords fewer difficulties than those before named, but in another way has others peculiar to itself...[T]hat bodies should be composed of numbers, and that this should be mathematical number, is impossible. For it is not true to speak of indivisible magnitudes; and however much there might be magnitudes of this sort, units at least have no magnitude; and how can a magnitude be composed of indivisibles? But arithmetical number, at least, consists of abstract units, while these thinkers identify number with real things; at any rate they apply their propositions to bodies as if they consisted of those numbers. *Metaphysics* 1083b8-1083b19

Here, Aristotle is expressing his misgivings about taking bodies to be composed wholly of arithmetical numbers units, suggesting that this is impossible. First, it is not true to speak of indivisible magnitudes, he says. Since a body is a magnitude (meaning continuous), it cannot be composed of indivisible magnitudes (such as the arithmetical units). This is an assertion he takes to have proven elsewhere. Secondly, on Aristotle's view geometrical units or points have no magnitude and so cannot be part of (or compose) a magnitude. I have already discussed all of this in the previous section. Surprisingly, Kirk G.S. and J.E. Raven (1957, 246ff) point out that it is *the Pythagoreans* who are confused.

> The unfortunate consequence of their diagrammatic representation of numbers was that the Pythagoreans, thinking of numbers as spatially extended and confusing the point of geometry with the unit of magnitude, tended to imagine both alike as possessing magnitude...It is true that Aristotle, in discussing the views of earlier thinkers, often confronts them with such logical consequences of their doctrines as they themselves never either enunciated or foresaw...[Aristotle]

> leaves no doubt that the Pythagoreans did indeed assume, that units are spatially extended; and when we come to consider the paradoxes of Zeno we shall find that it is against this assumption, along with the confusion of points and units, that they have their greatest force.

I disagree with Kirk and Raven's attribution of confusion to the Pythagoreans. It is *Aristotle* who is confused or misunderstood the upshot of Pythagorean number theory. Remember he said, "But how the first unit was constructed so as to have magnitude, [Pythagoreans] seem unable to say."(*Metaphysics* 1080b17-1080b21) Aristotle has no grounds for making this claim. We know from historians that all the mathematics we find in *The Elements* except for Book V was known before the time of Plato.[61] This mathematical knowledge includes the Pythagorean theory of proportions applicable to commensurables only, the discussion of arithmetical units and the mathematical knowledge in *The Elements* Book X on incommensurables. So we can reasonably expect the greatest student of Plato, Aristotle, to have known it. We may excuse Aristotle for being unfamiliar with the work of his contemporary Eudoxus, another student of Plato, who showed that magnitudes or bodies can be understood number theoretically according to the theory of proportions we find in *The Elements* Book V. But I think it is nothing short of confusion for Aristotle to base his objection to the Pythagoreans on the claim that the unit has magnitude. It is a confusion because according to the Pythagoreans the unit has no magnitude (in Aristotle's sense). It is a *number* that is the common measure of commensurable magnitudes (in Aristotle's sense). Here, I speculate: if Aristotle had been familiar with Eudoxus's work and been charitable to the Pythagoreans, he would not have insisted on his theory of potential infinite divisibility as the characteristic of the continuous – hence that magnitudes are infinitely divisible.

3.3 Magnitudes and Incommensurables

In the previous subsection I have defended the argument that the thesis of infinite divisibility is a result of two different conceptions of mathematics. I have still not discussed how the historical confusion regarding incommensurable magnitudes and infinite divisibility arose. Recall that one of the main arguments for infinite divisibility was the existence of incommensurable magnitudes. So now we must face two questions: (1) What are magnitudes? and (2) What are incommensurable magnitudes?

Earlier we saw that Aristotle distinguished magnitudes from numbers by saying that magnitudes are continuous and infinitely divisible. We remarked that this identification of

[61]See Heath (1981, Vol. 1, 216 - 217).

the continuous with the infinitely divisibility is a philosophical thesis that does not follow from the bisection theorem. Although *The Elements* identifies *arithmos* (number) with the collection of units in Book VII, it does not follow from *The Elements* alone that *megethos* (magnitude) is not composed of units, where "not composed of units" is the definition of continuous. It is, after all, open for someone to construe the "units" as *actual* points, not parts, of a dense point-set continuum (something which Berkeley does). Commentators and historians of mathematics have noticed that it is hard to grasp the meaning of *megethos* because *The Elements* does not give us a definition that tells us what magnitudes *are*.[62] What *The Elements* does give us is a theory of proportions, going back to the Pythagoreans and Eudoxus, that tells us at least how to deal with *megethos* rigorously. This is the account that we get in Books V, VII, and X. But in order to for me to show this and in order to understand Books V, VII, and Book X, we need to inquire into the incommensurables more closely.

Recall that Aristotle says that the Pythagoreans were unable to say how the unit was constructed so as to have magnitude. In order to evaluate Aristotle's claim, we need to look at how incommensurability was discovered, under what assumptions, and what conclusions the discoverers drew. There are three competing accounts: (1) the number-theoretic proof regarding the incommensurability of the diagonal of a square of unit length; (2) the proofs in Plato's dialogues and the method of finding the mean proportional between two plane similar numbers; (3) the method in *The Elements* Book X.[63] Let us look at these accounts in turn. I will not seek to disentangle which of these methods was the one that was actually used. Here it is a matter of speculation. For my purposes, the question I shall be seeking to answer is this: is the infinite divisibility of magnitudes assumed or does it follow from the given proof in the method?

1. *Number theoretic proof interpreted geometrically*

   The proof is familiar and proceeds by *reductio ad absurdum*. Let ABC be a right isosceles triangle with side of unit length. Suppose that the diagonal AC is commensurable to the side AB. Let $m : n$ be their ratio expressed in lowest terms (i.e. the greatest common divisor of m and n is 1). Now $AC^2 : AB^2 = m^2 : n^2$. Since $AC^2 = 2AB^2$ by the Pythagorean theorem (*The Elements* Book I. 47), it follows that $m^2 = 2n^2$. Hence m^2 is even and so is m. Since $m : n$ is in its lowest terms, it

[62] I will not attempt to speculate what Euclid meant by 'magnitude.' Here's where examples work better than definitions: lines, areas, volumes are magnitudes. See the discussion in Mueller (1981, 121f, 136 - 138) for an attempt to sort out what magnitudes are.

[63] These competing accounts are discussed in detail in Knorr (1975, 22 - 49) with references to Von Fritz (1945). See also Knorr (1981), Unguru (1977), and compare with Heath (1981, 202 - 209).

follows n is odd. Let $m = 2a$ for some a; then $4a^2 = 2n^2$ and $n^2 = 2a^2$, hence n is even. But this is impossible since n was shown to be odd. Therefore, the diagonal AC is incommensurable with the side AB.[64]

Let us waive the difficulty that this proof (Proposition 117 in Euclid Book X) was actually an interpolation as Heath (1956, Vol. 3, 2) suggests. The important point to take away from this proof is that it is number theoretic and nowhere in the proof has the assumption that finite lines are infinitely divisible entered into the reasoning. Aristotle was familiar with this proof as I've mentioned.[65] So it is unclear on what basis he concluded that magnitudes are infinitely divisible from this theorem. If this was indeed the way that incommensurables were shown to exist, then Berkeley is right to say that it is a *petitio principii* to conclude from this that finite lines are infinitely divisible.

2. *The proofs in Plato's dialogues and the method of finding the mean proportional*

   This number theoretic proof did not generalize in an obvious way to incommensurable square roots greater than $\sqrt{2}$. The proofs that $\sqrt{3}$, $\sqrt{5}$, ..., $\sqrt{17}$ are incommensurable with 1 as the unit of measure are reported in Plato's *Theaetetus*, where it is said they were developed by the Pythagorean Theodorus. There is some controversy regarding exactly how Theodorus proved these incommensurability results since Plato does not tell us the method. For this reason, Heath (1981, Vol. 1, 202 - 209) offers three hypotheses. (1) The method of successively approximating $\sqrt{3}$ by a geometric sequence with common ratio $\frac{1}{2}$; (2) the traditional number theoretic approach used to show that $\sqrt{2}$ is incommensurable with 1 as the unit of measure; and (3) a proposal by Zeuthen based on the method for detecting incommensurability given by Proposition 2 in Euclid Book X.[66] In any case, these are hypotheses and as far as I can tell, there is no mention of infinite divisibility in the proofs according to the methods suggested by these three hypotheses. In method (3) in particular, it is the existence of a non-terminating *number theoretic* process that tells us that we are dealing with incommensurable magnitudes. I have found no evidence in Berkeley that he is objecting to this non-terminating number theoretic process in the case of incommensurable magnitudes.

[64] See Heath (1981, 147 - 148) for discussion on how the Pythagoreans proved what is now Proposition 47 in *The Elements* Book I.

[65] See Heath (1956, Vol. 3, 2)

[66] Heath (1981, Vol. 1, 207) and Heath (1956, Vol. 3, 18) thinks that method (3) is similar to the Euclidean algorithm for finding the greatest common divisor. I return to a detailed discussion of this method in the next item.

According to historians (Heath, 1981, Vol. 1, 89), the mathematics in Plato's *Timaeus* has Pythagorean themes and contains references to the existence of a geometric mean between two square numbers and two geometric means between two cube numbers.[67] Barrow thought that the theorem proving the existence of a mean proportional between two square numbers was the basis of incommensurability and that the method presupposed the infinite divisibility of quantities. Here's how he puts it in *Mathematical Lectures* XV (my emphasis):

> The principal reason of incommensurability seems to be founded in this, that since a mean proportional number may always be found between two plane similar numbers because the product made by the multiplication of plane similar numbers is always a square number, whose root is that mean proportional ... since I say, things are thus in similar numbers, and it is demonstrated in the *Elements*, that it happens quite otherwise in all dissimilar numbers; there is no mean proportional number between two dissimilar plane numbers. [H]ence, if two quantities are supposed to be to one another in the [ratio] of two dissimilar numbers, and a mean proportional be found between those quantities, which may perpetually be done, *because of the indefinite divisibility of every quantity*, there will be no number in universal nature which can represent or answer to this quantity, and consequently, those being supposed and expressed by numbers, this will be incommensurable.

Barrow's point here sounds a lot more complicated than it is. It is actually Book VIII. Proposition 11.[68] Let's put his point in more modern terms. A plane number m is a number that is a product of two numbers a and b i.e. $m = ab$ (Book VIII. Proposition 5). According to Heath, plane similar numbers are what we call square numbers today. But it is possible to generalize plane similar numbers to include oblong (rectangular) numbers $m = ab$ and $n = cd$ such that the proportion $a : c :: b : d$ holds. Plane dissimilar numbers are oblong (rectangular) numbers $m = ab$ and $n = cd$ such that the proportion $a : c :: b : d$ does not hold.[69] The mean proportional number between two numbers m and n is what we call today the geometric mean of m, n. That

[67] See Heath (1956, 363)'s note to *The Elements* Book VIII Proposition 11

[68] Cf. Book X. Proposition 9.

[69] See Heath (1956, 293 - 294) commentary on *The Elements* Book VII, Def. 21. Compare with Book VI Proposition 13 (To two given straight lines to find a mean proportional) and the geometrico-algebraic method given in *The Elements* Book II Proposition 14 (To construct a square equal to a given rectilineal figure) involving the extraction of a square root.

is, the number x, such that $m : x :: x : n$. So $x = \sqrt{mn}$ which is distinguished from their arithmetic mean $\frac{m+n}{2}$. Barrow's point, following Book VIII. Proposition 11, is that there is a rational mean proportional *number* between two plane numbers m and n, just in case m and n are plane *similar* numbers. This is easy to see in the special case where m and n are square numbers since in that case $x = \sqrt{a^2b^2} = ab$. If m and n are plane dissimilar numbers, then in general $x = \sqrt{(ab) \cdot (cd)}$ is not a rational number. Barrow argues that this is the principal reason for incommensurability and that this follows because of the infinite (he uses the word 'indefinite') divisibility of every quantity. But nowhere in the proofs has the infinite divisibility of finite lines been assumed or concluded. So Berkeley is right that it is a petitio principii to conclude, on the basis of this argument, that finite lines are infinitely divisible.

3. *The Method of The Elements Book X Proposition 2*

This being said, there is a non-terminating method for detecting incommensurable magnitudes that is related to this method of finding the mean proportional.[70] Heath (1956, Vol. 3, 18) remarks that these propositions make essential use of the Euclidean division algorithm for finding the greatest common divisor between two numbers (I describe this method below). Von Fritz (1945) and Van Der Waerden (1961, 176f) call this method *anthyphairesis* and speculate that incommensurables were discovered by this method even though Heath (1981, Vol. 1, 207) finds it improbable. Let us call this method *epanalipsi-afairesis* (*repeated-subtraction*) in order to distinguish it from Aristotle's potential infinite divisibility.[71] Let's look at this method starting with Book X Proposition 2.

Book X Proposition 2

If, when the less of two unequal magnitudes is continually subtracted in turn from the greater, that which is left never measures the one before it, the magnitudes will be incommensurable.[72]

[70]See Knorr (1975, 29f) for discussion although even he thinks that it is very unlikely that this was how incommensurables were discovered.

[71]I did not know that this method had a name until I read a brief discussion in Furley (1967, 49) where he calls this process *antistrofi-afaireisis* (reciprocal-subtraction). I have chosen to call this process *epanalipsi-afairesis* (repeated-subtraction) in order to remain faithful or closer to the plain reading of the Greek text. In his discussion of the infinite in *Physics* III. 5 - 6 Aristole uses the term *division*(*diaresis*) most frequently as the antithesis of *addition*(*synthesis*). He occasionally speaks of *subtraction*(*afairesis*) and *diminution*(*kathairesis*). See also Heath (1956, Vol 1, 232). Could this be the method that presupposes or concludes that magnitudes are infinitely divisible?

[72]This proposition depends on **Book X Proposition 1**: Two unequal magnitudes being set out, if from the greater there be subtracted a magnitude greater than its half, and from that which is left a magnitude

Compare this with the number theoretic proposition in Book VII. There is a strong analogy although the one is about incommensurables and the other is about relative primes.

Book VII Proposition 1

Two unequal numbers being set out, and the less being continually subtracted in turn from the greater, if the number which is left never measures the one before it until a unit is left, the original numbers will be prime to one another.

The *epanalipsi-afairesis* method for detecting incommensurables is this: To determine the proportion between two lengths M and m *representing numbers*, of which M is the greater, first subtract m from M as many times as possible, leaving a remainder m'. Then subtract m' from m in the same way leaving a remainder m''. Then subtract m'' from m' and so on until no remainder (if at all) is left. The first length which can be subtracted thus without leaving any remainder is *the unit* in terms of which the ratio $M : m$ can be expressed. The unit will vary according to what these lengths M and m are. These units are not geometrical points but numerical measures.

Assuming that this was how incommensurables were first detected (and it is reasonable to do so since Book X is largely due to Theaetetus and predates Aristotle), then what the Pythagoreans called a unit (*monad*) is what we call today 1 (if the numbers are relatively prime from Book VII.1) or the greatest common divisor of two composite numbers if it existed (from Book VII.2). This "unit" (of measure) can be used to measure (*metron*) the magnitude i.e. how great (*to pelikos*) a homogeneous quantity is relative to another homogeneous quantity. If the answer to the question "How great?" could be expressed as a *ratio* (i.e. it is *logos*) or proportion, then the *numbers* were rational and the magnitudes representing them were commensurable. The answers which the Pythagoreans would give would always be in terms of proportions, $4 : 2 :: 2 : 1$ which means that 4 is 2 times as great as 2 using 2 as the unit. If there is no greatest common divisor (including 1) between two numbers, then the two magnitudes representing them are incommensurable. There is no common measure or no way of comparing them with respect to size (by Book V. Definitions 3 and 4). This would be the case if the process of *epanalipsi-afairesis* did not terminate after a finite number of steps. But it is one thing to say that this non-terminating number theoretic process is true for incommensurable magnitudes and it is another thing to

greater than its half, and if this process be repeated continually, there will be left some magnitude which will be less than the lesser magnitude set out. This is a version of the Eudoxus-Archimedes Axiom in Book V. Definition 4.

conclude or assume on the basis of this, that the continuity of magnitudes consists in their being infinitely divisible.

To see why this confusion was a mistake and misleading, consider a line equal in length with the circumference of a circle and a line equal in length with the diameter of the same circle. It is common ground between the Pythagoreans and Aristotle that both these lines are continuous. Suppose that there are numbers which can be represented by these lengths, say 22 (the circumfrence) and 7 (the diameter). There is no greatest common divisor between these numbers (since this is one approximation of the constant π). The process of *epanalipsi-afairesis* does not terminate in the case of these two numbers and many others like them. But this has nothing to do with continuity or infinite divisibility of the lengths representing these numbers as Aristotle thought. Thus, incommensurability does not show that the essence of the continuity of magnitudes is infinite divisibility. Rather, it shows that there are pairs of magnitudes *representing numbers* for which this process of *epanalipsi-afairesis* does not terminate after a finite number of steps.

Of some of these incommensurables, there are those that cannot be represented as a ratio between known magnitudes (they are *alogos*, inexpressible or irrational, because of this; rational otherwise). Notice that even though the magnitude (*megethos*) representing $\sqrt{2}$ is incommensurable using 1 as the unit of measure, the number (*arithmos*) $\sqrt{2}$ is not irrational or *alogos* in the Pythagorean sense. $\sqrt{2}$ can be expressed as the ratio between known magnitudes, namely, the ratio between the diagonal of a right-isosceles triangle with side of unit and one of its sides. So incommensurability does not imply irrationality. This is how what we mean by irrational numbers today differs from how the Pythagoreans conceived of them. However, magnitudes representing numbers such as $\sqrt{19}$ are not only incommensurable with 1 as the unit of measure but also irrational. Thus irrationality implies incommensurability. I am not sure how to think of π in Pythagorean terms. It seems to me that even though π is incommensurable with 1 as the unit of measure, it is not irrational in the Pythagorean sense since it can be expressed as the ratio between the circumference of a circle and its diameter.

4 Conclusion

It is the conflation of the Aristotelian thesis of infinite divisibility with the non-terminating *epanalipsi-afairesis* characteristic of incommensurability that has stayed with mathematicians and philosophers for millennia. This is the ancient prejudice that Berkeley was alluding to. If the method that was first used for detecting incommensurable magnitudes representing numbers besides '$\sqrt{2}$' was indeed Book X Proposition 2, then one way to read Book X is as a geometric (hence rigorous) translation or formulation of number theoretic *facts*. Incommensurability arises when number theoretic facts are being embedded in geometry, for example by trying to find the ratio or proportion between two magnitudes that represent certain numbers. This raises the question of the proper foundations for mathematics: is it geometry or arithmetic? If Van der Waerden is right, then according to the Pythagoreans, the way to place their number theoretic investigations on rigorous foundations was to cash them out geometrically. But this should therefore be a caution of trying to interpret what Pythagoreans took to be a number theoretic fact (a non-terminating process) as evidence for a geometrical fact (the infinite divisibility of finite lines). Barrow, in his mathematical lectures (Lecture III), famously argued for the identity of geometry with arithmetic. Similarly, Aristotle objected to the importation of the Pythagorean number theoretic discussion of units *that can be represented geometrically* as spatially extended magnitudes. He raised the valid question, "If a unit is indivisible, how can it be spatially extended?" To be sure, this is the right question for Aristotle to ask given his conception of mathematics; since for him spatially extended parts of magnitudes are divisible ad infinitum. But the Pythagoreans meant something completely different when they spoke of representing units as magnitudes in geometry. We've seen that for the Pythagoreans, these units are *units of measure* of the ratio between magnitudes i.e. what we refer to today as either 1 (for two relatively prime numbers) or the greatest common divisor of two magnitudes (if it existed) of two composite numbers. These units have nothing to do with the divisibility (or lack thereof) of magnitudes since continuity was a common notion.

In conclusion, let me recapitulate the main points of this paper. I have given evidence that the theory of proportions was motivated by number theoretic discoveries. The theory of proportions developed by Eudoxus and Theaetetus was given in order to place these discoveries on a rigorous foundation in *The Elements*'s Books V and X. Besides dissociating the bisection theorem with infinite divisibility, I given reasons for resisting the assimilation of a number theoretic process with the geometric thesis of infinite divisibility; and the Aristotelian identification of continuity with infinite divisibility. Thus, Berkeley is right in pointing out that infinite divisibility of finite lines is not an axiom or theorem in *The Elements*. If Aristotle's view – that *actual* indivisible points cannot compose a magnitude – assumes the philosophical thesis of infinite divisibility, then I hope to have shown that this thesis flows out of a different conception of mathematics and is not necessary to develop the theory of proportions along Pythagorean lines and hence to handle magnitudes (or continuous quantities).

This then opens up the possibility of thinking in terms of indivisible points with position as the actual constituents of magnitudes – a possibility that was of course taken up by later mathematicians like Dedekind and Cantor. Berkeley, in fact, was more favorably disposed to indivisibles than to the theory of differential and integral calculus based on infinitesimals. It was this method of indivisibles which began to suggest to Berkeley the possibility of geometric minima i.e. the conception of magnitudes as composed of actual indivisible points. But this is the subject for my other paper. What I have done in this paper is substantiate Berkeley's claim that infinite divisibility of finite lines is neither an axiom nor a theorem in *The Elements* but is rooted in Aristotelian prejudice.

References

Aristotle (1984). *Complete Works of Aristotle Revised Oxford Translation. Edited by Jonathan Barnes. 2 vols.* Princeton University Press.

Arnauld, Antoine and Nicole, Pierre (1996). *Antoine Arnauld and Pierre Nicole: Logic or the Art of Thinking (Translated by JV. Buroker).* Cambridge University Press.

Arthur, R. T. W. (2019). Book Review of Geoffrey Hellman and Stewart Shapiro. Varieties of Continua: From Regions to Points and Back. *Philosophia Mathematica 27*, 148 – 152.

Barrow, I. (1734). *The Usefulness of Mathematical Learning Explained and Demonstrated: Being Mathematical Lectures Read in the Publick Schools at the University of Cambridge (Translated by Rev. John Kirkby).* Frank Cass & Co. Ltd. Original date of publication is 1683.

Black, M. (1967). Achilles and the Tortoise. In Wesley C. Salmon (Ed.), *Zeno's Paradoxes*, pp. 67 – 81.

Diogenes Laertius (2018). *Lives of the Eminent Philosophers (Translated by Pamela Mensch).* Oxford University Press.

Ehrlich, P. (2014). An Essay in Honor of Adolf Grünbaum's Ninetieth Birthday: A Reexamination of Zeno's Paradox of Extension. *Philosophy of Science 81*(4), 654–675.

Fano, V., F. Orilia, and G. Macchia (2014). *Space and Time: A Priori and A Posteriori Studies.* Walter de Gruyter.

Fogelin, R. (1988). Hume and Berkeley on the Proofs of Infinite Divisibility. *The Philosophical Review 97*(1), 47–70.

Franklin, J. (1994). Achievements and Fallacies in Hume's Account of Infinite Divisibility. *Hume Studies 20*(1), 85–101.

Friedman, M. (1995). Matter and Material substance in Kant's Philosophy of Nature: The Problem of Infinite Divisibility. In *Proceedings of the Eighth International Kant Congress*, Volume 1, pp. 595–610.

Furley, D. J. (1967). *Two Studies in the Greek Atomists.* Princeton University Press.

Garber, D. (1992). *Descartes' Metaphysical Physics.* University of Chicago Press.

Gow, J. (1968). *A Short History of Greek Mathematics.* Chelsea Publishing Company.

Heath, T. (1956). *The Thirteen Books of Euclid's Elements. 3 Vols.* Dover Publications.

Heath, T. (1981). *A History of Greek Mathematics (2 Vols.).* Dover Publications.

Holden, T. (2002). Infinite Divisibility and Actual Parts in Hume's Treatise. *Hume Studies 28*(1), 3–25.

Jacquette, D. (1996). Hume on Infinite Divisibility and Sensible Extensionless Indivisibles. *Journal of the History of Philosophy 34*(1), 61–78.

Jesseph, D. M. (1993). *Berkeley's Philosophy of Mathematics.* The University of Chicago Press.

Jesseph, D. M. (2005). Berkeley's Philosophy of Mathematics. In K. P. Winkler (Ed.), *The Cambridge Companion to Berkeley*, pp. 266 – 310. Cambridge University Press.

Kanamori, A. (2020). Cantor and Continuity. To appear in a philosophical volume on the concept of continuity edited by Stewart Shapiro, Cambridge University Press, 2020.

Keill, J. (1745). *An Introduction to Natural Philosophy: Or, Philosophical Lectures Read in the University of Oxford, A.D. 1700 (4th Edition).* M. Senex, W. Innys, T. Longman and T. Shewell. Original date of publication is 1700.

Kirk G.S. and J.E. Raven (1957). *The Presocratic Philosophers.* Cambridge University Press.

Kline, M. (1972). *Mathematical Thought from Ancient to Modern Times (3 Vols.)*. Oxford University Press.

Knorr, W. R. (1975). *The Evolution of The Euclidean Elements: A Study of the Theory of Incommensurable Magnitudes and Its Significance for Early Greek Geometry*. D. Reidel Publishing Company: Dordrecht-Holland.

Knorr, W. R. (1981). Aristotle and Incommensurability: Some Further Reflections. *Archive for History of Exact Sciences*, 1–9.

Lear, J. (1982). Aristotle's Philosophy of Mathematics. *The Philosophical Review 91*(2), 161–192.

Linnebo, Ø. and S. Shapiro (2019). Actual and Potential Infinity. *Noûs 53*(1), 160–191.

Luce, A. A. and T. E. Jessop (1948 – 1957). *The Works of George Berkeley, Bishop of Cloyne*. Thomas Nelson Ltd.

Maddy, P. (1997). *Naturalism in Mathematics*. Oxford University Press.

Manchak, J. and B. W. Roberts (2016). Supertasks. In E. N. Zalta (Ed.), *The Stanford Encyclopedia of Philosophy* (Winter 2016 ed.). Metaphysics Research Lab, Stanford University.

Mancosu, P. (1996). *Philosophy of Mathematics and Mathematical Practice in the Seventeenth Century*. Oxford University Press.

Miller, Jr., F. D. (1982). Aristotle against the Atomists. In N. Kretzmann (Ed.), *Infinity and Continuity in Ancient and Medieval Thought*, pp. 87 – 111. Cornell University Press.

Mueller, I. (1981). *Philosophy of Mathematics and Deductive Structure in Euclid's Elements*. MIT Press.

Mwakima, D. (2020). Berkeley's Mathematical Atomism. unpublished work in progress.

Pfeiffer, C. (2018). *Aristotle's Theory of Bodies*. Oxford University Press.

Pressman, H. M. (1997). Hume on Geometry and Infinite Divisibility in the Treatise. *Hume Studies 23*(2), 227–244.

Proclus (1970). *Proclus: A Commentary on the First Book of Euclid's Elements (Trans. Glenn R. Morrow)*. Princeton University Press.

Reeder, P. (2018). Labyrinth of Continua. *Philosophia Mathematica 26*(1), 1–39.

Schechtman, A. (2019). Three Infinities in Early Modern Philosophy. *Mind 128*(512), 1117–1147.

Sherry, D. M. (1988). Zeno's Metrical Paradox Revisited. *Philosophy of Science 55*(1), 58–73.

Simplicius (2014). *Simplicius On Aristotle Physics 6 (Translated by David Konstan).* Bloomsbury Publishing.

Skyrms, B. (1983). Zeno's Paradox of Measure. In *From Zeno to Arbitrage: Essays on Quantity, Coherence, and Induction*, pp. 5 – 37. Oxford University Press.

Sorabji, R. (1982). Atoms and Time Atoms. In N. Kretzmann (Ed.), *Infinity and Continuity in Ancient and Medieval Thought*, pp. 17 – 86. Cornell University Press.

Unguru, S. (1977). Incommensurability and Irrationality: A New Historical Interpretation. *History of Science 15*, 216 – 227.

Van Der Waerden, B. L. (1961). *Science Awakening (English translation by Arnold Dresden with additions of the author).* Oxford University Press.

Von Fritz, K. (1945). The Discovery of Incommensurability by Hippasus of Metapontum. *Annals of Mathematics*, 242–264.